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Dispersion of tracers in the deep ocean

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Exact solutions are used to contrast the behaviour of vertical dye streaks and of point discharge in the deep ocean. After a few inertial periods the growth of the contaminant cloud in both cases can be modelled by a diffusion equation with slowly varying coefficients. However, the apparent long-term diffusivity for the dye streak can be as much as three times as large as the horizontal diffusivity for the point discharge, depending upon the precise time of release. This anomalously large rate of spreading for the dye streak can persist for several weeks until sufficient time has elapsed for vertical diffusion over the vertical lengthscale of the dominant inertial-frequency waves.

1. Introduction

The possible use of the sea bed for the disposal of high-level radioactive waste has led to increased interest in the mechanics of contaminant dispersion in the sea. Indeed, a deep-ocean tracer-release experiment (DOTREX) is being planned. One facet of the dispersion problem is to quantify the great disparity between the horizontal and vertical rates of spreading (Ewart & Bendiner 1981).

The prevalent internal waves and inertial-frequency oscillations in the deep ocean distort a contaminant cloud back and forth about its averaged shape. If at the extreme stages of such a distortion material is diffused out of the cloud, then on return to the mean state the 'lost' material will be displaced relative to the main body of the cloud. Thus there is an augmented rate of spreading due to the currents. This shear-dispersion mechanism was first identified by Townsend (1951), and made more widely known by Taylor (1953). A qualitative explanation for the larger horizontal than vertical rate of spreading in the deep ocean is simply that the oscillatory currents, and hence the displacements, are predominantly horizontal.

A particular velocity field that is amenable to mathematical analysis is when the velocity varies linearly with height. Okubo (1967) gives solutions for the first few horizontal moments of the concentration distribution for a point discharge of dye. Unfortunately, because of the vertical averaging process, these global moments can be quite misleading (see §5). This same linear velocity profile has been studied by Young, Rhines & Garrett (1982), who derive the concentration distribution for a vertical dye streak. A disconcerting feature of their solution is that the effective long-term horizontal diffusivity can vary by a factor of 3, depending upon the precise time of release. Such intrinsic variability of the horizontal rate of spreading would greatly reduce the usefulness of any single dye-release experiment in the ocean.

The primary purpose of the present paper is to derive the concentration distribution for a point discharge of dye. As was correctly surmised by Young *et al.* (1982), there is no longer a strong memory effect and the horizontal shear-dispersion coefficient is equal to the minimum of the range of values for a vertical dye streak. A secondary purpose of this paper is to clarify the nature of the memory effect for a vertical dye streak. It is shown that the anomalously large rate of horizontal spreading can persist for several weeks, until sufficient time has elapsed for vertical diffusion over the vertical lengthscale of the oscillatory shear.

2. Advection–diffusion equation in distorted axes

The stable vertical stratification of the oceans means that the steady and unsteady flow velocities are predominantly horizontal, with horizontal lengthscales greatly in excess of the vertical lengthscales. Thus we model the advection–diffusion equation for the contaminant concentration $c(x, z, t)$:

$$\partial_t c + u(z,t)\,\partial_x c - K_1\,\partial_x^2 c - K_3\,\partial_z^2 c = 0, \tag{2.1}$$

where u is the horizontal velocity, K_1 the horizontal diffusion coefficient, and K_3 the vertical diffusivity. (Strictly, the use of diffusivities for a turbulent fluid is justifiable only if the space- and timescales on which we are studying the dispersion process exceed the corresponding scales for the turbulence. Furthermore, (2.1) does not provide any information concerning concentration fluctuations, which can be vitally important in the context of high-level radioactive waste products.)

Close to the level of discharge ($z = 0$) we can approximate the velocity profile by a linear shear

$$u = u_0(t) + z\alpha(t). \tag{2.2}$$

The linear shear (2.2) tilts particles back and forth. Thus the shape of the dye cloud can be expected to share at least some of the tilting. Hence we introduce the distorted horizontal coordinate

$$X = x - \int_{t_0}^{t} u_0(t')\,dt' - zG(t, t_0), \tag{2.3}$$

where t_0 is the time of discharge, and the distortion factor G remains to be chosen. Careful application of the change-of-variable rules for partial differentiation leads to the transformed equation

$$\partial_t c + (\alpha - \partial_t G)\,z\partial_X c - [K_1 + G^2 K_3]\,\partial_X^2 c + 2GK_3\,\partial_X\,\partial_z c - K_3\,\partial_X^2 c = 0. \tag{2.4}$$

For clarity we have restricted attention to unidirectional flows. The extension to include a second horizontal coordinate is given in §6.

3. Young, Rhines & Garrett's solution

If the initial conditions are independent of z then the choice

$$G = \int_{t_0}^{t} \alpha(t')\,dt' \tag{3.1}$$

leads to the time-dependent diffusion equation

$$\partial_t c - [K_1 + K_3 G^2]\,\partial_X^2 c = 0. \tag{3.2}$$

For a vertical dye streak the solution is

$$c = \frac{q \exp\left(-\tfrac{1}{2}X^2/\sigma^2\right)}{\sigma(2\pi)^{\frac{1}{2}}}, \tag{3.3}$$

with

$$D_1 = \frac{1}{2}\frac{d\sigma^2}{dt} = K_1 + K_3\left[\int_{t_0}^{t} \alpha(t')\,dt'\right]^2 \tag{3.4}$$

(Young *et al.* 1982, equations (8), (9)).

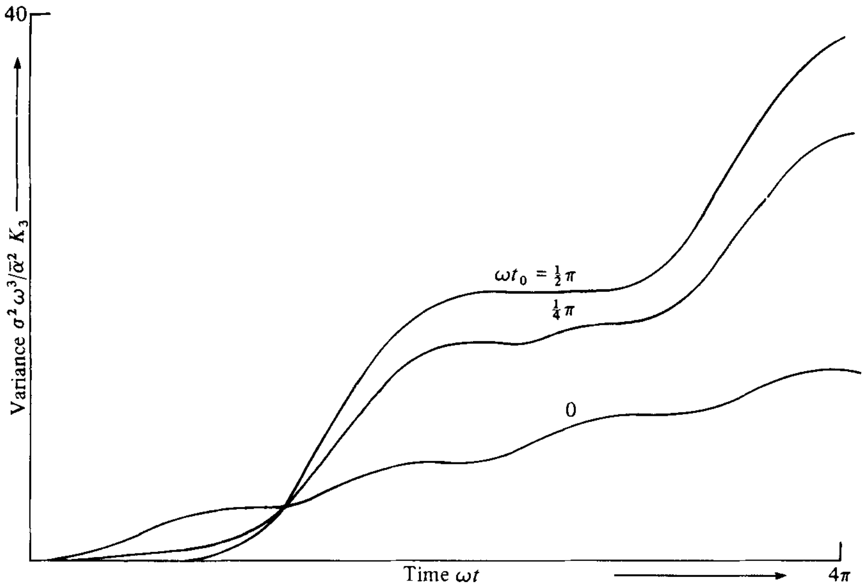


FIGURE 1. The horizontal variance as a function of time for a vertical dye streak released at times $\omega t_0 = 0, \frac{1}{4}\pi, \frac{1}{2}\pi$ in an oscillatory linear shear with $(\bar{\alpha}/\omega)^2 = 20K_1/K_3$.

For a steady current, with α constant, (3.4) yields the result

$$\sigma^2 = 2K_1(t-t_0)+\tfrac{2}{3}\alpha^2 K_3(t-t_0)^3. \tag{3.5}$$

In this case the variance never settles down to a linear growth rate, so the dispersion cannot be modelled in terms of a horizontal diffusion equation.

If the velocity shear is sinusoidal in time,

$$\alpha(t) = \bar{\alpha}\cos\omega t, \tag{3.6}$$

then the effective horizontal diffusivity (3.4) is given by

$$D_1 = K_1+\frac{1}{2}\left(\frac{\bar{\alpha}}{\omega}\right)^2 K_3\{1+2\sin^2\omega t_0-4\sin\omega t_0\sin\omega t-\cos 2\omega t\}. \tag{3.7}$$

Figure 1 shows the evolution of σ^2 for several different discharge times t_0.

At large times after discharge we can ignore the oscillatory contribution to D_1, because the oscillations in σ^2 become dominated by the systematic long-term growth. From (3.7) we find that the effective diffusion coefficient $\langle D_1\rangle$ comprises a diffusive term and a shear term:

$$\langle D_1\rangle = K_1+\frac{1}{2}\left(\frac{\bar{\alpha}}{\omega}\right)^2 K_3\{1+2\sin^2\omega t_0\}. \tag{3.8}$$

In oceanic conditions the tilt $\bar{\alpha}/\omega$ of particle surfaces can be quite large (Gargett *et al.* 1982). Thus there is indeed a great disparity between the horizontal and vertical rates of spreading $\langle D_1\rangle$ and K_3. As remarked in §1, there is a factor-of-three variability in the shear term depending upon the initial phase of the oscillatory current when the dye streak was released.

For a random current there is a similar persistent memory of the initial conditions.

We introduce a reference time $\tilde{t}$ such that on average the subsequent displacements are symmetrical:

$$\langle \tilde{G} \rangle = \left\langle \int_{\tilde{t}}^{t} \alpha(t')\,dt' \right\rangle = 0. \tag{3.9}$$

Thus we can decompose the distortion factor G into a symmetric part and a memory part:

$$G = \tilde{G} - \int_{\tilde{t}}^{t_0} \alpha(t')\,dt'. \tag{3.10}$$

The resulting formula for the long-term effective diffusivity is

$$\langle D_1 \rangle = K_1 + K_3 \langle \tilde{G}^2 \rangle + K_3 \left[\int_{\tilde{t}}^{t_0} \alpha(t')\,dt' \right]^2. \tag{3.11}$$

The final memory term can be thought of as being the effect in the horizontal plane of the vertical diffusion of the mean tilted z-structure of the tracer distribution.

4. Solution for a point discharge of dye

Townsend (1951) noted that for a point release in a time-dependent linear shear flow the contaminant distribution is exactly Gaussian. Thus, if the distortion factor G is correctly chosen, we can then expect the advection–diffusion equation (2.4) to have a solution

$$c = \frac{Q \exp\left(-\tfrac{1}{2}X^2/V_1 - \tfrac{1}{2}z^2/V_3\right)}{2\pi V_1^{\frac{1}{2}} V_3^{\frac{1}{2}}}, \tag{4.1}$$

where $V_1(t, t_0)$, $V_3(t, t_0)$ are the horizontal and vertical variances.

The direct substitution of the proposed solution (4.1) into (2.4) generates terms of the forms X^2c, z^2c, Xzc, c. The first three of these classes of terms yield the respective equations

$$D_1 = \frac{1}{2}\frac{dV_1}{dt} = K_1 + K_3 G^2, \tag{4.2}$$

$$K_3 = \frac{1}{2}\frac{dV_3}{dt}, \tag{4.3}$$

$$\frac{dG}{dt} = \alpha - \frac{2K_3 G}{V_3}, \tag{4.4}$$

while the fourth class of terms is a linear combination of (4.2) and (4.4).

For a point discharge the vertical variance has precisely the pure diffusive value

$$V_3 = 2K_3(t - t_0), \tag{4.5}$$

and the corresponding solution for the distortion factor is

$$G = \int_{t_0}^{t} \frac{t' - t_0}{t - t_0}\,\alpha(t')\,dt'. \tag{4.6}$$

A crucial difference from the vertical-dye-streak result (3.1) is the fading memory $(t' - t_0)/(t - t_0)$. Thus at large times after discharge the distortion of the contaminant cloud becomes independent of the initial conditions.

If the velocity shear is sinusoidal,

$$\alpha = \bar{\alpha} \cos \omega t, \tag{4.7}$$

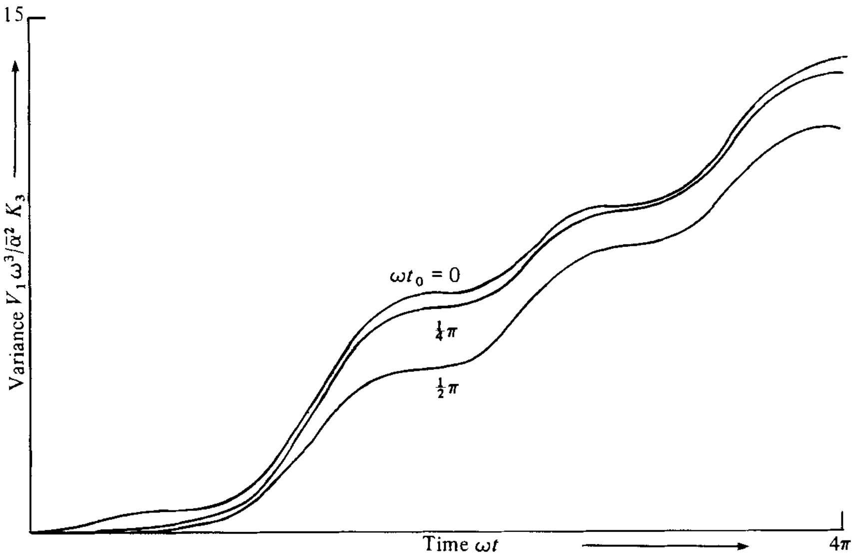


FIGURE 2. The horizontal variance as a function of time for a point discharge of dye released at times $\omega t_0 = 0, \frac{1}{4}\pi, \frac{1}{2}\pi$ in an oscillatory linear shear with $(\bar{\alpha}/\omega)^2 = 20K_1/K_3$.

then the effective horizontal diffusivity (4.2) is given by

$$D_1 = K_1 + \frac{1}{2}\left(\frac{\bar{\alpha}}{\omega}\right)^2 K_3\left\{1 - \cos 2\omega t + \frac{4\sin\omega t[\cos\omega t - \cos\omega t_0]}{\omega(t-t_0)} + \frac{2[\cos\omega t - \cos\omega t_0]^2}{\omega^2(t-t_0)^2}\right\}. \tag{4.8}$$

Figure 2 shows the evolution of V_1 for several different discharge times t_0. After a few flow periods the oscillations in V_1 become dominated by the long-term growth. The effective long-term diffusion coefficient $\langle D_1\rangle$ is given by

$$\langle D_1\rangle = K_1 + \frac{1}{2}\left(\frac{\bar{\alpha}}{\omega}\right)^2 K_3. \tag{4.9}$$

As surmised by Young *et al.* (1982), this is equal to the minimum value for the vertical-dye-streak problem (3.8). Moreover, there is no longer the persistent-memory effect. The diffusivity is merely a function of the fluid properties (K_1, K_3) and of the wave field $(\bar{\alpha}, \omega)$.

5. Horizontal moments for a point release

Although he was aware that a complete solution could be found, Okubo (1967) chose to calculate the horizontal moments of the concentration distribution. We can recover his main results by vertically averaging the exact results (4.1).

To do this we first note that the quadratic decay exponent has the alternative forms

$$\frac{\left[x - \int_{t_0}^{t} u_0(t')\,dt' - zG\right]^2}{2V_1} + \frac{z^2}{2V_3} = \frac{\frac{1}{2}\left[x - \int_{t_0}^{t} u_0(t')\,dt'\right]^2}{V_1 + G^2V_3} + \frac{1}{2}\left[z - \frac{\left(x - \int_{t_0}^{t} u_0(t')\,dt'\right)GV_3}{V_1 + G^2V_3}\right]^2 \frac{V_1 + G^2V_3}{V_1V_3}. \tag{5.1}$$

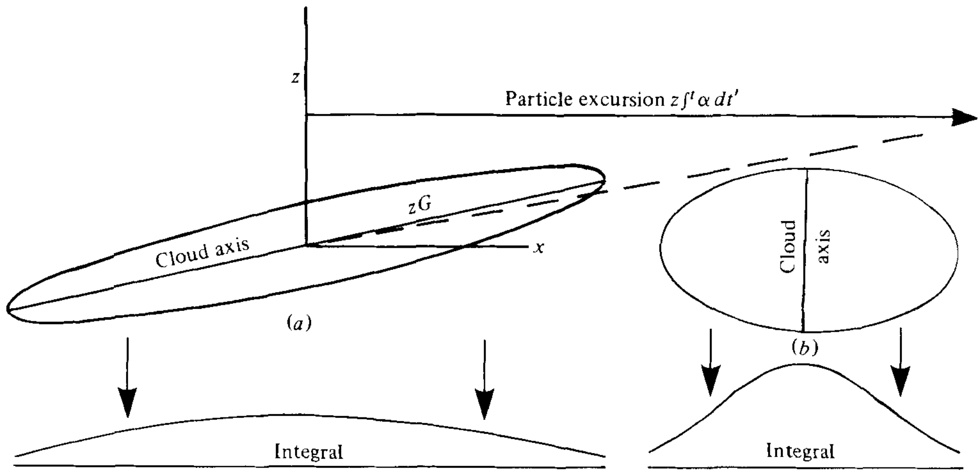


FIGURE 3. The shape of a contaminant cloud (for a point discharge) and its vertical integral when (a) the particle displacement is at a maximum, and (b) when it is zero.

Similarly, the denominator in the exact solution (4.1) has the alternative forms

$$2\pi V_1^{\frac{1}{2}} V_3^{\frac{1}{2}} = 2\pi (V_1 + G^2 V_3)^{\frac{1}{2}} \left[\frac{V_1 V_3}{V_1 + G^2 V_3} \right]^{\frac{1}{2}}. \tag{5.2}$$

Thus we can infer that in any vertical section the concentration distribution (4.1) is Gaussian, and the vertically integrated concentration $\|c\|$ is given by the further Gaussian expression

$$\|c\| = Q \exp \left\{ - \frac{\frac{1}{2}\left[x - \int_{t_0}^{t} u_0(t')\, dt' \right]^2}{V_1 + G^2 V_3} \right\} \frac{1}{2\pi (V_1 + G^2 V_3)}. \tag{5.3}$$

The pivoting of the velocity profile about the discharge height $z = 0$ has the consequence that vertical integrating totally obscures the back-and-forth tilting of the contaminant distribution (see figures 3*a*, *b*). Thus the vertically integrated solution (5.3) has a centre of mass that simply moves with the velocity $u_0(t)$ at the discharge height. Moreover, instead of the actual horizontal variance V_1, the expression (5.3) for $\|c\|$ has variance $V_1 + G^2 V_3$. For a sinusoidal current (4.7) the distortion factor G, given by (4.6), is likewise sinusoidal. This leads to the asymptotic expressions

$$V_1 \sim 2K_1(t - t_0) + \left(\frac{\bar{\alpha}}{\omega} \right)^2 K_3(t - t_0), \tag{5.4}$$

$$V_1 + G^2 V_3 \sim 2K_1(t - t_0) + 2\left(\frac{\bar{\alpha}}{\omega} \right)^2 K_3(t - t_0) - \left(\frac{\bar{\alpha}}{\omega} \right)^2 K_3(t - t_0) \cos 2\omega t \tag{5.5}$$

(see figure 4). The large oscillatory term in (5.5) would suggest that at no stage could the dispersion be modelled by a diffusion equation (Okubo 1967, equation (10)). Also, when the tilt $\bar{\alpha}/\omega$ is large, the linear trend is twice the non-averaged result (5.4).

For steady currents the vertically averaged moments are equally misleading. With α constant, the two expressions for the horizontal variance are

$$V_1 = 2K_1(t - t_0) + \tfrac{1}{6}\alpha^2 K_3 (t - t_0)^3, \tag{5.6}$$

$$V_1 + G^2 V_3 = 2K_1(t - t_0) + \tfrac{2}{3}\alpha^2 K_3 (t - t_0)^3, \tag{5.7}$$

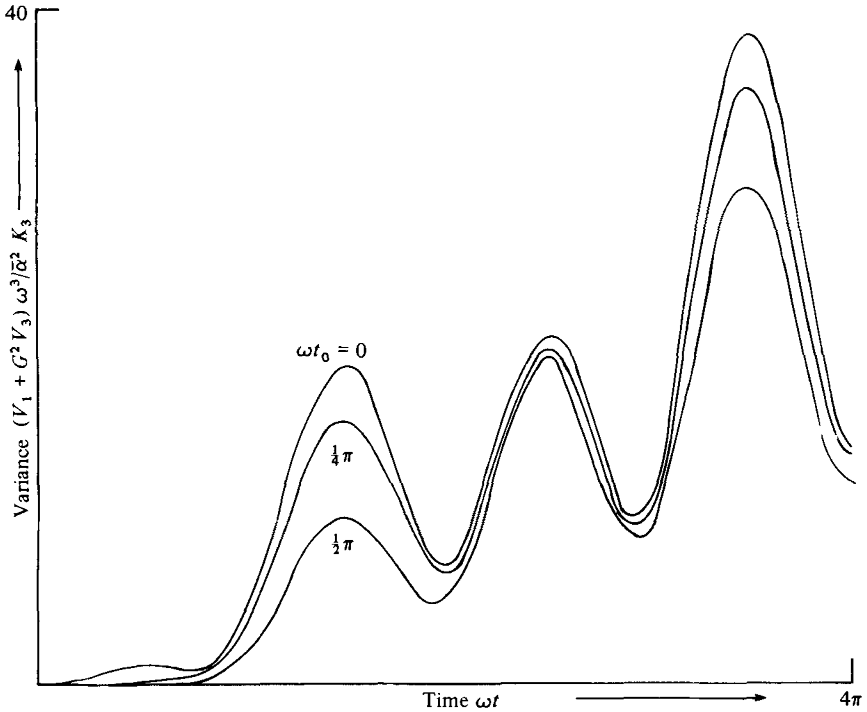


FIGURE 4. The horizontal variance of the vertically integrated concentration as a function of time for a point discharge of dye released at time $\omega t_0 = 0$, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$ in oscillatory linear shear with $(\bar{\alpha}/\omega)^2 = 20K_1/K_3$.

with a factor-of-four difference in the shear dispersion. It happens that (5.7) is the same as the result (3.5) for a vertical dye streak. Surprisingly, Okubo (1967, equation (10)) did not note the disparity between his vertically averaged result (5.7) and the non-averaged result (5.6) first derived by Carter & Okubo (1965).

6. A three-dimensional Gaussian solution for a point release

If the results are to be applied to real oceanic conditions (Young *et al.* 1982), then one complication that has to be dealt with is that the oscillatory currents are two-dimensional. This is intrinsically the case for the circular particle displacements associated with inertial frequency oscillations.

Thus, we generalize the preliminary analysis of §2 to incorporate a y-component of velocity:

$$v = v_0(t) + \alpha_2(t)\,z, \tag{6.1}$$

$$Y = y - \int_0^t v_0(t')\,dt' - zG_2(t), \tag{6.2}$$

$$\begin{aligned}\partial_t c + (\alpha_1 - \partial_t G_1)\,z\partial_X c + (\alpha_2 - \partial_t G_2)\,z\partial_Y c - [K_1 + G_1^2 K_3]\,\partial_X^2 c - 2G_1 G_2 K_3 \partial_X \partial_Y c \\ - [K_1 + G_2^2 K_3]\,\partial_Y^2 c + 2G_1 K_3 \partial_X \partial_z c + 2G_2 K_3 \partial_Y \partial_z c - K_3 \partial_z^2 c = 0.\end{aligned} \tag{6.3}$$

Here (α_1, α_2) are the two components of the velocity shear, (G_1, G_2) are the two components of the coordinate distortion, and we have assumed that the diffusion K_1 is the same in both horizontal directions.

As noted by Townsend (1951), the Gaussianity of the solution for a point source is not dependent upon the number of dimensions. In the horizontal plane the principal axes are not necessarily aligned along the coordinate axes. Thus we pose the solution

$$c = \frac{Q \exp(-P)}{(2\pi)^{\frac{3}{2}} V_1^{\frac{1}{2}} V_2^{\frac{1}{2}} V_3^{\frac{1}{2}}}, \tag{6.4}$$

with
$$P = \frac{(X\cos\phi + Y\sin\phi)^2}{2V_1} + \frac{(Y\cos\phi - X\sin\phi)^2}{2V_2} + \frac{z^2}{2V_3}, \tag{6.5}$$

where $\phi(t)$ is the changing orientation of the principal axes.

The counterparts of equation (4.1) are:

$$D_1 = \frac{1}{2}\frac{dV_1}{dt} = K_1 + K_3(G_1\cos\phi + G_2\sin\phi)^2, \tag{6.5}$$

$$D_2 = \frac{1}{2}\frac{dV_2}{dt} = K_1 + K_3(G_2\cos\phi - G_1\sin\phi)^2, \tag{6.6}$$

$$\frac{d\phi}{dt} = \frac{2K_3(G_1\cos\phi + G_2\sin\phi)(G_2\cos\phi - G_1\sin\phi)}{V_1 - V_2}, \tag{6.7}$$

while the results (4.5), (4.6) are unchanged:

$$V_3 = 2K_3 t, \quad G_1 = \int_0^t \frac{t'}{t}\alpha_1(t')\,dt', \quad G_2 = \int_0^t \frac{t'}{t}\alpha_2(t')\,dt'. \tag{6.8), (6.9), (6.10}$$

Again, we emphasize that the t'/t factors imply that at moderately large times the distortion factors become independent of the time of discharge. Thus there is not any persistent memory effect at large times after discharge.

As an illustrative example we take the horizontal particle displacements to be elliptical with principal axis at angle ψ and with ellipticity e:

$$\alpha_1 = \bar{\alpha}[\cos\psi\cos\omega t + (1-e^2)\sin\psi\sin\omega t], \tag{6.11}$$

$$\alpha_2 = \bar{\alpha}[\sin\psi\cos\omega t - (1-e^2)\cos\psi\sin\omega t). \tag{6.12}$$

Restricting our attention to large times after discharge, we have

$$G_1 \sim \frac{\bar{\alpha}}{\omega}[\cos\psi\sin\omega t - (1-e^2)\sin\psi\cos\omega t], \tag{6.13}$$

$$G_2 \sim \frac{\bar{\alpha}}{\omega}[\sin\psi\sin\omega t + (1-e^2)\cos\psi\cos\omega t]. \tag{6.14}$$

Tentatively, we suppose that the contaminant concentration ellipse becomes aligned with the flow ellipse, i.e. $\phi \sim \psi$. Thus at large times we have

$$D_1 \sim K_1 + \frac{1}{2}\left(\frac{\bar{\alpha}}{\omega}\right)^2 K_3\{1 - \cos 2\omega t\}, \tag{6·15}$$

$$D_2 \sim K_1 + \frac{1}{2}\left(\frac{\bar{\alpha}}{\omega}\right)^2 K_3(1-e^2)^2\{1 + \cos 2\omega t\}, \tag{6.16}$$

$$\frac{d\phi}{dt} \sim \sin 2\omega t\left(\frac{\bar{\alpha}}{\omega}\right)^2 \frac{K_3(1-e^2)}{V_1 - V_2}. \tag{6.17}$$

Unless the ellipticity is zero, the contaminant cloud grows more rapidly in the principal than in the transverse direction. Hence $V_1 - V_2$ grows linearly with time, and therefore the angle ϕ oscillates with ever-decreasing amplitude. Thus it is indeed consistent to assume that ϕ asymptotes to the assumed orientation ψ. The principal values of the effective long-term horizontal dispersion coefficient are related to the respective magnitudes of the shears in the two directions:

$$\langle D_1\rangle = K_1 + \frac{1}{2}\left(\frac{\bar{\alpha}}{\omega}\right)^2 K_3, \tag{6.18}$$

$$\langle D_2\rangle = K_1 + \frac{1}{2}\left(\frac{\bar{\alpha}}{\omega}\right)^2 (1-e^2)^2 K_3. \tag{6.19}$$

In the limiting case of a circular current, we have $\langle D_1\rangle = \langle D_2\rangle$, so the contaminant cloud becomes circular, and the angle ϕ ceases to be of significance.

7. A point release in a general shear

Another complication of the real oceans is that the internal waves have all vertical lengthscales and may be comparable with the vertical size of the dye cloud. Thus the effective horizontal velocity $u_0(t)$ and the effective linear shear rate $\alpha(t)$ may differ from the obvious Taylor-series definitions. For clarity we shall revert to the two-dimensional case with flow confined to the x-direction.

The Gaussian solution (4.1) can be regarded as being the first term in a two-dimensional Hermite series expansion (Chatwin & Sullivan 1981). In view of the simple x-independent structure of the flow field, it is convenient to deal with one dimension at a time. At each level we take the centroid position to be at $x = \chi(z,t)$, and the variance to be $V_1(z,t)$. Thus we pose the Hermite series (Smith 1982, equation (2.2)):

$$c = Q\left\{a^{(0)}(z,t) + \sum_{n=3}^{\infty} \frac{a^{(n)}(z,t)\, He_n(X/V_1^{\frac{1}{2}})}{V_1^{\frac{1}{2}n}}\right\} \frac{\exp(-\frac{1}{2}X^2/V_1)}{(2\pi)^{\frac{1}{2}}\, V_1^{\frac{1}{2}}}, \tag{7.1}$$

with
$$X = x - \chi(z,t), \quad \chi = V_1 = 0 \quad \text{at} \quad t = t_0. \tag{7.2}$$

The coefficients of the first few Hermite polynomials in the advection–diffusion equation (2.1) yield the equations

$$\partial_t a^{(0)} - K_3\, \partial_z^2 a^{(0)} = 0, \tag{7.3}$$

$$\partial_t \chi - \frac{2K_3\, \partial_z a^{(0)}}{a^{(0)}}\, \partial_z \chi - K_3\, \partial_z^2 \chi = u(z,t), \tag{7.4}$$

$$\partial_t V_1 - \frac{2K_3\, \partial_z a^{(0)}}{a^{(0)}}\, \partial_z V_1 - K_3\, \partial_z^2 V_1 = 2K_1 + 2K_3(\partial_z \chi)^2. \tag{7.5}$$

Subsequent equations in this sequence are given by Smith (1982, equation (2.7)). Here it suffices to note that the higher-order coefficients $a^{(n)}$ are very small.

For a point release the solution of (7.3) is

$$a^{(0)} = \frac{\exp(-\frac{1}{2}z^2/V_3)}{(2\pi V_3)^{\frac{1}{2}}}, \tag{7.6}$$

with
$$V_3 = 2K_3(t-t_0). \tag{7.7}$$

Thus we can evaluate the quotient

$$-\frac{2K_3\,\partial_z a^{(0)}}{a^{(0)}} = \frac{z}{t-t_0}. \tag{7.8}$$

Immediately after discharge it is this $z/(t-t_0)$ factor in the equation (7.4) for the centroid that makes the memory fade so rapidly, on a time scale of just one wave period.

At large times we can ignore the $z/(t-t_0)$ factor, and the equations (7.4), (7.5) for the centroid and the horizontal variance become exactly the same as for a vertical dye streak:

$$\partial_t \chi - K_3\,\partial_z^2 \chi = u(z,t), \tag{7.9}$$

$$\partial_t V_1 - K_3\,\partial_z^2 V_1 = 2K_1 + 2K_3(\partial_z \chi)^2. \tag{7.10}$$

In particular, for a sinusoidal velocity field

$$u = \frac{\bar{\alpha}}{m}\sin mz\cos\omega t, \tag{7.11}$$

the equilibrium solutions of (7.9), (7.10) are

$$\chi = \frac{\bar{\alpha}}{m}\sin mz\frac{\omega\sin\omega t + m^2K_3\cos\omega t}{\omega^2+m^4K_3^2}, \tag{7.12}$$

$$\langle D_1\rangle = \tfrac{1}{2}\langle\partial_t V_1\rangle = K_1 + \frac{K_3\bar{\alpha}^2}{2(\omega^2+m^4K_3^2)}. \tag{7.13}$$

Hence, for an internal wave spectrum there is a cut-off

$$m < \left(\frac{\omega}{K_3}\right)^{\frac{1}{2}} \tag{7.14}$$

to the range of vertical wavenumbers that contribute to the shear dispersion process (Young *et al.* 1982).

8. A vertical dye streak in a sinusoidal shear

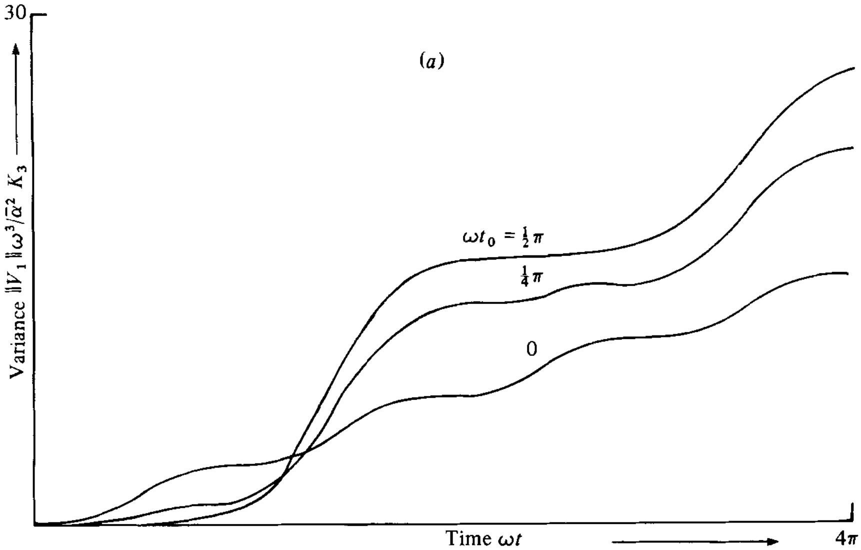
For a vertical dye streak the memory timescale for the centroid position and for the variance is of order $1/m^2K_3$. Thus in the real ocean the memory paradox discussed in §3 persists until sufficient time has elapsed for vertical diffusion over the vertical lengthscale of the internal waves.

As a specific example, we again use the sinusoidal velocity profile (7.11). The exact solution of (7.9) for the centroid displacement of an initially vertical dye streak is

$$\chi = \frac{\bar{\alpha}}{m}\sin mz\frac{\omega\sin\omega t + m^2K_3\cos\omega t}{\omega^2+m^4K_3^2} - \frac{\bar{\alpha}}{m}\sin mz\exp\left[-m^2K_3(t-t_0)\right]\frac{\omega\sin\omega t_0 + m^2K_3\cos\omega t_0}{\omega^2+m^4K_3^2}. \tag{8.1}$$

The corresponding solution for the variance comprises a constant and a $\cos 2mz$ component. The time-dependent diffusivity $\|D_1\|$ associated with the vertically averaged variance is given by

$$\|D_1\| = \tfrac{1}{2}\partial_t\|V_1\| = K_1 + \frac{F(t,t_0)\,\tfrac{1}{2}K_3\bar{\alpha}^2}{\omega^2+m^4K_3^2}, \tag{8.2}$$



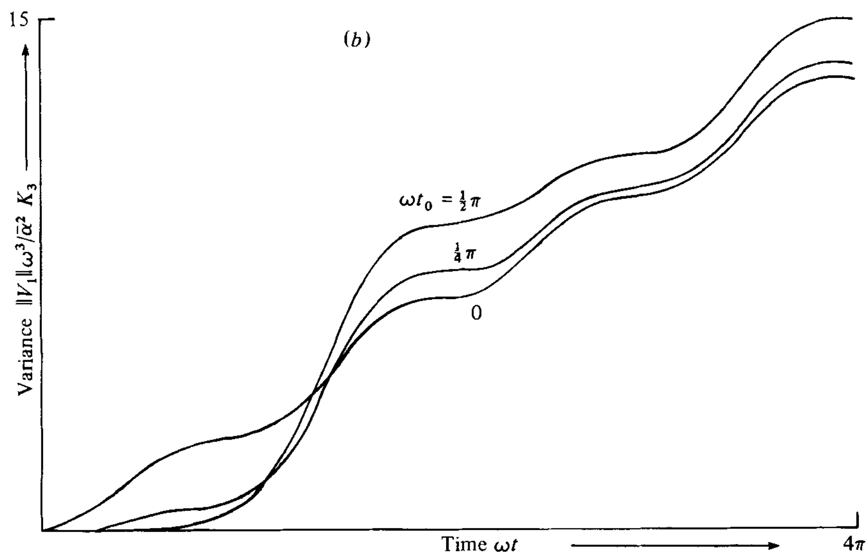


FIGURE 5. The vertically averaged horizontal variance as a function of time for a vertical dye streak released at times $\omega t_0 = 0, \frac{1}{4}\pi, \frac{1}{2}\pi$ in an oscillatory sinusoidal shear with $(\bar{\alpha}/\omega)^2 = 20K_1/K_3$ and with vertical wavenumber (*a*) $m = \frac{1}{4}(\omega/K_3)^{\frac{1}{2}}$; (*b*) $m = \frac{1}{2}(\omega/K_3)^{\frac{1}{2}}$.

where the memory effects are incorporated into the time-dependent function $F(t, t_0)$:

$$F = 1+2\sin^2(\omega t_0+\theta)\exp(-2m^2K_3(t-t_0))-\cos(\omega t+2\theta)$$
$$-4\exp(-m^2K_3(t-t_0))\sin(\omega t+\theta)\sin(\omega t_0+\theta), \quad (8.3)$$

with
$$\sin\theta = \frac{m^2K_3}{(\omega^2+m^4K_3^2)^{\frac{1}{2}}}. \quad (8.4)$$

It is the first two non-oscillatory terms that show how the effective long-term horizontal diffusion coefficient for a vertical dye stream gradually reduces to the value (7.13) appropriate for a point release.

The intrinsic variability of the apparent rate of horizontal diffusion for a vertical dye streak is down to about 30% after a time of $1/m^2K_3$ (see figures 5*a*, *b*). For example, if in m.k.s. units we take

$$K_3 = 5\times 10^{-5}, \quad m = 0{\cdot}1 \tag{8.5}$$

(Ewart & Bendiner 1981; Gargett *et al.* 1981), then our estimate of this memory timescale is 2×10^6 s, or about 23 days. The DOTREX experiment is envisaged as having a duration of about a month in the first instance. Thus we conclude that a point release is very much to be preferred to a vertical streak if the memory effect is to be avoided.

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